Safe and Effective Determinant Evaluation

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The problem:

given $n \times n$ matrix A, with b-bit entries,
is $\det A > 0$?

comparisons : sorting

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det signs : geometric algorithms

Need det with low *relative* error:

In geometric computations,
numerical tests yield combinatorial objects.
Bad tests can yield wildly wrong results.

Prior (theoretical) approaches:

- exact arithmetic
  - can be slow: apparently nb bits needed.
- change the algorithm
  - results for few problems
  - hard to do

Approach here:
exact answers for det sign,
using both approx. and exact arithmetic.

Result

Let A be an $n \times n$ integer matrix
with b-bit entries.

The sign of $\det A$ can be computed
using arithmetic of precision

$$\beta + b + 1.5n + 1$$

in $O(n^3 b)/\beta$ time, $\beta > 0$.

Moreover

- $O(n^3)$ time for unit-cost b-bit ops;
- running time dependent on $\lg \mathcal{OD}\, A$;
- $O(n^3)$ time for $\lg \mathcal{OD}\, A/\beta = O(1)$;

The algorithm is:

- similar to Lovász's for basis reduction;
- solves a simpler problem, and so
- has better bounds.

Orthogonalization

The general approach:

- massage the matrix to good condition
  - preserve the determinant
  - orthogonal matrices
    (pairwise $\perp$ columns)
    are perfectly conditioned
- apply Gaussian elimination.

Elementary column ops preserve the det,
So:
apply these exactly
to make A “nearly” orthogonal.

Gram-Schmidt Orthogonalization

Let $a/b \equiv \frac{a \cdot b}{b \cdot b}$. Then $b \perp a - (a/b)b$.

Given $A = [a_1 \cdots a_n]$,
G-S computes orthogonal
$c_1 \ldots c_n$ so that
$\text{span}\{c_1 \ldots c_k\} = \text{span}\{a_1 \ldots a_k\}$

That is,

$$a_k = c_k + \sum_{j \leq k} (a_k/c_j) c_j$$

for $k := 1$ **upto** n **do**
$c_k := a_k$;
for $j := k - 1$ **downto** 1 **do** $c_k -= c_j(c_k/c_j)$;

In approximate arithmetic,
lose sign when $\|a_k\| \gg \|c_k\| \approx 0$;

Gram-Schmidt for conditioning

Maintain $b_j \approx c_j$ for $j < k$
using G-S with approximate ops.

Modify a_k using *exact* ops:

for $j := k - 1$ **downto** 1 **do** $a_k -= a_j \lceil a_k/b_j \rfloor$;

Have

$$a_k = c_k + \sum_{j \le k} (a_k/c_j) c_j$$

and $a_k/c_j \le\approx 1/2$ for $j < k$.

Not close to orthogonal if $\|a_k\| \gg \|c_k\|$.

Gram-Schmidt with scaling

Small c_k is not a problem for integer A: scale up a_k exactly and reduce again.

while $\|b_k\| > \|a_k - b_k\|$ {
    choose integer s;
    $a_k \mathrel{*}= s$;
    for $j := k-1$ **downto** 1 **do** $a_k \mathrel{-}= a_j \lceil a_k/b_j \rfloor$;
}

Choose s as large as possible, subject to bound on entries of A.

Since the entries of A are **integers**, small magnitude $\Rightarrow$ small representation.

Conclusions

- also useful for:
  - obtaining stable algorithms;
  - equations of hyperplanes;
  - basis reduction?

- novelties:
  - determinants discounted in NA lit;
  - inequality bounds;
  - forward error analysis;

(concluding Conclusions)

- what about:
  - practice?
  - rational matrices?
  - widely varying row sizes?
  - lower bounds for Gaussian elimination?