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A Las Vegas Algorithm for Linear Programming When the Dimension is Small

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Outline

- Results
- The smallest enclosing sphere
  - the algorithm
  - analysis of the algorithm
- Modifications for LP
- Conclusions

Problem: linear programming with
n inequality constraints in d variables,

- $O(n2^{2^d})$ [Meg];
- $O(n3^{d^2})$ [C][D];
- $O(nd^{3d+\epsilon})$ Las Vegas [DF];

New bound:
$O(d^2 n) + O(d \log n) O(d)^{d/2+O(1)}$
expected arithmetic operations, Las Vegas.

Time bound from
$O(d \log n)$ calls to simplex,
on subproblems with $\approx d^2$ constraints.

The smallest enclosing sphere

Given
$S \subset E^d$ of size n, in general position,
find $B^*(S)$,
the smallest closed ball containing S.

Facts: $B^*(S)$ exists, is unique,
and there exists $S^* \subset S$ with
$|S^*| \leq d + 1$ and $B^*(S^*) = B^*(S)$.

The algorithm

The general idea: focus in on S^* using random sampling.

choose random $R \subset S$ with $|R| = 2(d+1)^2$;
compute $B^*(R)$ using some algorithm;
Let $V \leftarrow S \setminus B^*(R)$;

Fact: V contains at least one point of S^* (unless $B^*(R) = B^*(S)$ so $V = \phi$)

Otherwise $S^* \subset B^*(R)$,
with $B^*(R)$ no larger than $B^*(S)$;
So $B^*(R) = B^*(S^*) = B^*(S)$;

Fact: The expected size of V is $(d+1)n/r = n/2(d+1)$.

So V is a small set that must contain a member of S^*.

An iterative algorithm

$w_p \leftarrow 1$ for all $p \in S$;
repeat
  choose (weighted) random $R \subset S$;
  $V \leftarrow S \setminus B^*(R)$;
  $w_p \leftarrow 2w_p$ for all $p \in V$;
until $V = \phi$

Compare the total weight of S
with the total weight of S^*;

Each time,
some point in S^* is in V.

After $k(d+1)$th step, weight of S^* is at least $(d+1)2^k$;

Each iteration, expected weight of V is dW/r
W is the current weight, at first $W = n$.

The next weight is $W(1+(d+1)/r)$;
After $k(d+1)$th step,
$W \leq n(1+(d+1)/r)^{k(d+1)} \approx ne^{k/2}$.

With $k = O(\log n)$, algorithm will stop.

Proof that the expected size of V is $(d+1)n/r = \sqrt{n}$:

Let $\mathcal{F}_S$ denote $\{B^*(T) \mid T \subset S\}$
For $B \in \mathcal{F}_S$, let T_B be the smallest $T \subset S$ with $B = B^*(T)$. Then $|T_B| \leq d+1$ for $B \in \mathcal{F}_S$.

Let $\mathcal{F}_S^j = \{B \in \mathcal{F}_S \mid j = |S \setminus B|\}$.
Similarly define $\mathcal{F}_R$, $\mathcal{F}_R^j$.

Note that $\mathcal{F}_S^0 = \{B^*(S)\}$.
Fact: $|\mathcal{F}_S^1| \leq d+1$.

Proof of claim:

Let $I_B = 1$ when $B = B^*(R)$, 0 otherwise.

The expected number of points outside $B^*(R)$ is

$$E\left[\sum_{\substack{j\geq 0\\ B\in\mathcal{F}_S^j}} I_B j\right] = \sum_{\substack{j\geq 0\\ B\in\mathcal{F}_S^j}} E[I_B]j = \sum_{\substack{j\geq 0\\ B\in\mathcal{F}_S^j}} \text{Prob}\{B = B^*(R)\}j$$

$$= \sum_{\substack{j\geq 0\\ B\in\mathcal{F}_S^j}} \binom{n-j-|T_B|}{r-|T_B|} j \Big/ \binom{n}{r}$$

$$\leq \frac{n-r+d+1}{r-d-1} \sum_{\substack{j\geq 0\\ B\in\mathcal{F}_S^j}} \binom{n-j-|T_B|}{r-|T_B|-1} j \Big/ \binom{n}{r}$$

$$\leq (n/r) \sum_{j\geq 0, B\in\mathcal{F}_S^j} \text{Prob}\{B \in \mathcal{F}_R^1\}$$

$$= (n/r)E[|\mathcal{F}_R^1|] = \frac{n}{r}(d+1).$$

Modifications for LP

max$\{cx \mid Ax \leq b\}$,
where A is $n \times d$

a similar algorithm, except:

- sample constraint inequalities, not points
- modify to assure feasibility
- give answers for unbounded subproblems
- break ties by choosing shortest optimal point
- use simplex for base case

Concluding Remarks

- What about small n/d?
- Any implications for simplex?
- Any implications for combinatorial problems?